ECE 4390 Engineering Computations IV

### Laboratory 3: Finite Difference Methods for
Electrostatics and Transmission Lines

Tuesday, October 18, 2011   **Due Date: Tuesday, November 1, 2011**

Part I: Solution of Laplace's Equation Using Finite Differences

A)   For the two dimensional rectangular region shown in the figure write a program to solve for the scalar potential $\phi(x, y)$ on a rectangular grid using Successive Over-Relaxation (SOR).
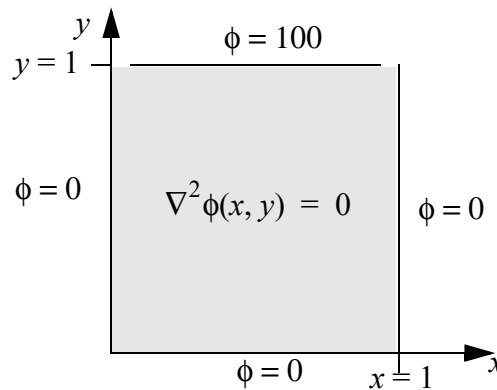

Use the relative displacement norm, ε, to stop the iterative loop. The relative displacement norm, ε, at iteration $(m+1)$ is derived from the displacement norm δ as:

$$\delta = \sum_{i=1}^{N} \left| \phi_i^{(m+1)} - \phi_i^{(m)} \right| = \|\Delta\phi\|, \ \varepsilon = \frac{\delta}{\|\phi\|} = \delta / \sum_{i=1}^{N} \left| \phi_i^{(m+1)} \right|$$

where N is the total number of grid points in the two dimensional grid.

B)   Using $h = \Delta x = \Delta y = 0.1$ determine the optimum over-relaxation constant ω_{opt}. Choose $\varepsilon = 10^{-6}$ as the stopping condition. Plot number of iterations vs. ω. Plot the potential using a contour plot with 10 level lines between 0 and 100 V. Have your program calculate the total electric flux emanating from the enclosed region by numerically integrating the normal component of the electric field around the boundary. Discuss why this should be so.

C)   Decrease h by a factor of 2 three times to $h = \Delta x = \Delta y = \{0.05, 0.025, 0.0125\}$ and set $\varepsilon = 10^{-6}$. Compare the required number of iterations for each h using the optimum over-relaxation constant ω_{opt} found in part B.

D)   Reformulate the problem taking advantage of symmetry across the line defined by $x = 0.5$. What boundary conditions are required for the new problem and what is the finite difference boundary condition you will use? Modify your computer program for this new problem and check that the results are the same.

E)   Modify your program to solve for the p.u.l. capacitance of the transmission line shown in the figure.
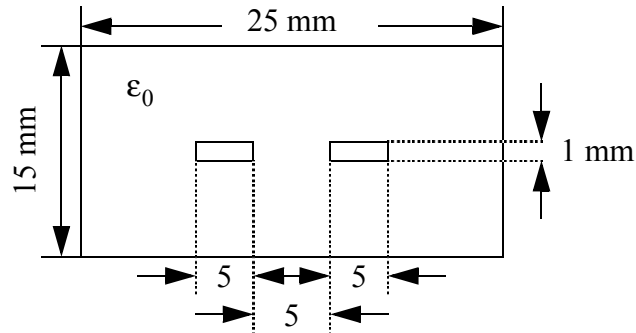
F)   Derive a finite difference approximation which can be used for a 2-D problem having dielectric discontinuities as shown in the figure. Give the finite difference formula for the computational molecule shown (see Sadiku's book).
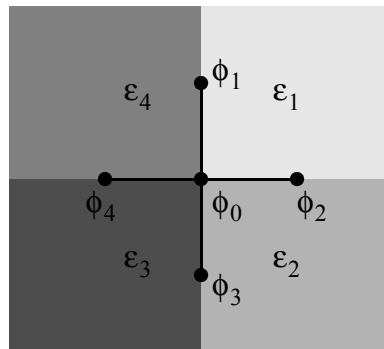
Modify your program to take into account different regions of the dielectric constant and find the p.u.l capacitance for the same structure as above but now with a dielectric layer as shown.
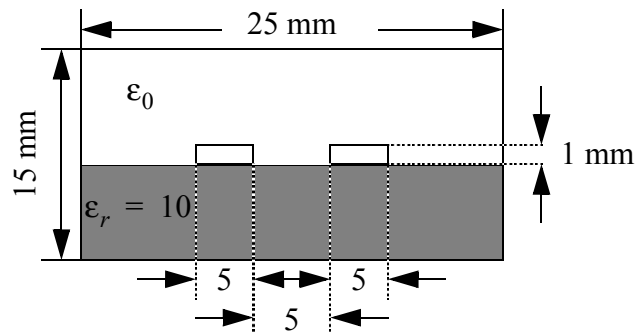
Part I: Solution of the Transmission Line Equations

For a transmission line with series per-unit-length (p.u.l.) resistance, R [Ω/m]; series p.u.l. inductance, L [H/m]; shunt p.u.l. capacitance, C [F/m]; and shunt p.u.l. conductance, G [S/m] the governing equations are given

$$\begin{cases} \dfrac{\partial}{\partial x}v(x,\,t) + L\dfrac{\partial}{\partial t}i(x,\,t) + Ri(x,\,t) = 0 \\[2mm] \dfrac{\partial}{\partial x}i(x,\,t) + C\dfrac{\partial}{\partial t}v(x,\,t) + Gv(x,\,t) = 0 \end{cases}$$

G) Write a program to calculate the voltage and current along a transmission line ($0 \le x \le l$) using the interlaced Leap-Frog scheme.

Test your program for a lossless transmission line of length $l = 1$ m, which has p.u.l. capacitance of $C = 50$ pF/m and a p.u.l. inductance of $L = 0.5$ μH/m. Assume that line is short-circuited at the $x = 0$ end and open-circuited at the $x = l$ end. What is the velocity, c_0, of propagation for this line?

Assume that the initial conditions are $v_0(x) = 10(u(x-0.45) - u(x+0.55))$ where $u(x)$ is the unit step-function, and plot the voltage and current along the line at the following times: 2.2, 2.3, 2.5, 3.5 ns. Run your program for the time step equal to 0.9, 1.0, and 1.01 times the Courant limit $\Delta t = \Delta x / c_0$. (Choose whatever spatial discretization Δx you deem appropriate.) Compare to the exact solution on the same plot.

H) Modify your program to handle lossy lines.

Test your program for the same transmission line example above but add a p.u.l. resistance of 0.1 [Ω/m] and a p.u.l conductance of 0.001 [S/m]. Run your code again at the Courant limit for each of the times given in question A.

I) Modify your program to deal with resistive Thèvenin source terminations at either end. The Thèvenin voltage source should be a Gaussian waveform specified as

$$v_T(t) = Ae^{-((t-t_0)/\sigma_0)^2}$$

where A, t_0, σ_0 are input parameters, and the Thèvenin resistance is R_T.

Test your program for the same transmission lines given above with a termination $R_T = 50$ Ω at the $x = 1$ end and with a Thèvenin voltage source have a resistance of $R_T = 100$ Ω at the $x = 0$ end. Experiment with different parameters for the Gaussian waveform which allow you to clearly see the reflection coefficients at bothe ends.